CHAPTER 9: CIRCLES

YOUR ULTIMATE GUIDE TO MASTERING THE FOUNDATIONS OF MATHEMATICS

1. Introduction: Circles in Real Life and Mathematics

Circles are one of the most fundamental and pervasive shapes in nature and human design. From clock faces and coins to wheels and planetary orbits, circles surround us. Understanding the geometry of circles helps solve many practical problems and builds foundational mathematical intuition.

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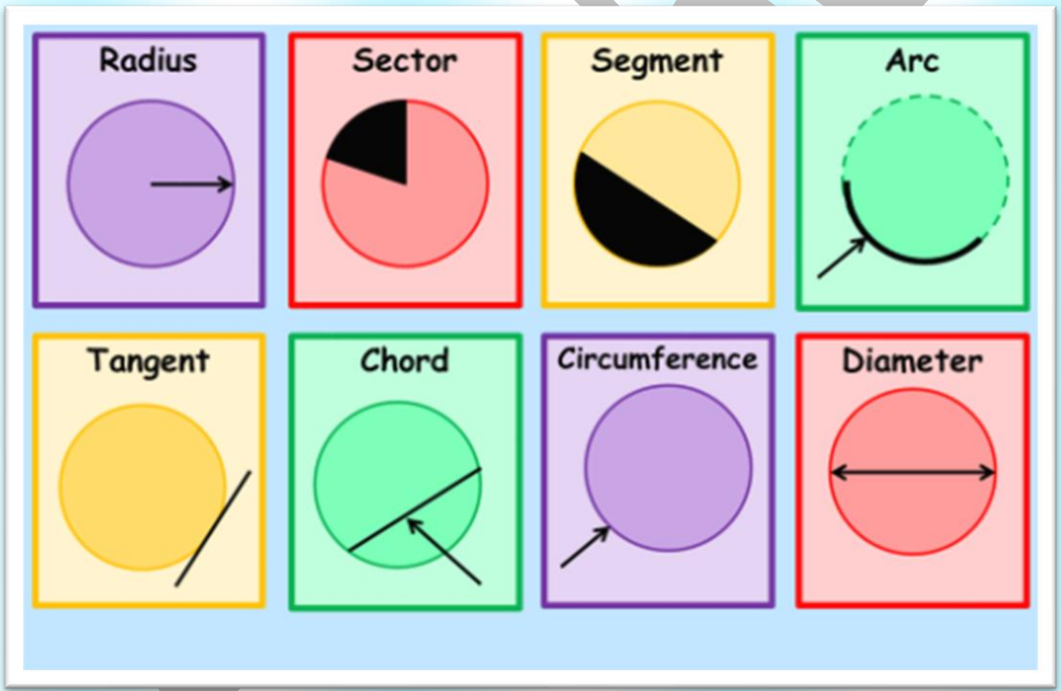
2. Definitions and Key Terms with Detailed Explanation and Visuals

- Circle: The set of all points in a plane at a fixed distance (called the radius) from a fixed point (called the centre). Denoted as circle with centre O.
- Radius (r): Any segment OA where O is centre and A lies on the circle. All radii of a circle are equal.

- Diameter (d): The chord passing through the centre. It is the longest chord of a circle and equals twice the radius.
- Chord: Any segment joining two points A and B on the circle.
- Arc: A continuous portion of the circumference between two points.
- Sector: The region bounded by two radii OA, OB and the arc AB.
- Segment: The area enclosed by a chord AB and the corresponding arc AB.
- Circumference: The total length around the circle;



mathematically $C = 2\pi r$.

- Tangent: A line touching the circle at exactly one point T. The radius OT is perpendicular to the tangent.
- Secant: A line cutting the circle at two points.

3. Circle Properties and Formulas — Stepwise Deduction

3.1 Length of an Arc

The length of an arc corresponding to central angle θ degrees is:

$$l = \frac{\theta}{360^\circ} \times 2\pi r$$

Derivation: **Using the proportion of the angle θ to complete 360°, derive arc length relative to circumference.**

3.2 Area of Sector

The area enclosed by two radii and arc (sector) with angle θ:

$$A_{\text{sector}} = \frac{\theta}{360^\circ} \times \pi r^2$$

3.3 Area of Segment

Area of segment = Area of sector – Area of triangle formed by radii.

- **To find area of triangle OAB, use the formula for triangle area with two sides and included angle:**

$$A_{\triangle} = \frac{1}{2} r^2 \sin\ \theta$$

where θ is angle AOB in radians or degrees.

Hence:

$$A_{\text{segment}} = A_{\text{sector}} - A_{\triangle} = \frac{\theta}{360^\circ}\pi r^2 - \frac{1}{2}r^2 \sin\,\theta$$

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4. Important Theorems with Proofs

Theorem 1: Equal chords subtend equal angles at the center

- **If chords AB and CD are equal, then $\angle AOB = \angle COD$.**

Proof:

- **In $\triangle OAB$ and $\triangle OCD$,**
- **$OA = OC, OB = OD$ (radii),**
- **$AB = CD$ (given),**
- **By SSS congruence, $\triangle OAB \cong \triangle OCD$,**
- **Therefore, $\angle AOB = \angle COD$.**

Theorem 2: Perpendicular from center to chord bisects the chord

- **If OM is perpendicular to chord AB, then AM = MB.**

Proof:

- **Triangles OMA and OMB are right-angled and share side OM,**
- **$OA = OB$ (radii),**
- **So, by RHS congruence, $\triangle OMA \cong \triangle OMB$,**

- **Thus AM = MB.**

Theorem 3: Angle subtended by arc at center is twice the angle subtended at circumference

Proof:

1. Draw a circle with center O.

2. Arc AB subtends ∠AOB at the center and ∠ACB at the circumference.

3. Join OA, OB, and OC.

4. Triangles OAC and OBC are isosceles (OA = OC, OB = OC).

5. Let ∠OAC = α and ∠OBC = β.

6. Then ∠ACB = $\alpha + \beta$.

7. Also, ∠AOB = $2\alpha + 2\beta = 2(\alpha + \beta)$.

8. Therefore, ∠AOB = 2 × ∠ACB.

Other Theorems:

1. **If the angles subtended by two chords at the centre are equal, then the chords are equal.**
2. **The line drawn through the centre to bisect a chord is perpendicular to the chord.**

3. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).
4. Chords equidistant from the centre of a circle are equal in length.
5. Angles in the same segment of a circle are equal.
6. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on a circle (concyclic points).
7. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
8. If the sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.

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5. Important Solved Examples

Example 1: Finding radius when chord length and distance from center are given

- Chord length $AB = 8\ cm$
- Distance from center to chord $OM = 3\ cm$
- Find radius OA

Solution:

- AB bisected at M, $AM = 4\ cm$
- Right triangle OMA with right angle at M:

$$OA^2 = OM^2 + AM^2 = 3^2 + 4^2 = 9 + 16 = 25$$
$$OA = 5\ cm$$

Example 2: Find length of tangent from external point

- Given $PT = 12\ cm$, radius $r = 5\ cm.$

Solution:

$$OP = \sqrt{PT^2 + r^2} = \sqrt{144 + 25} = 13\ cm$$

Example 3: Prove that angle in same segment are equal

Solution:

1. Draw a circle with center O.

2. Let arc AB subtend ∠ACB and ∠ADB at points C and D on the same segment.

3. Join OA and OB.

4. By the angle at the center theorem, ∠AOB = 2∠ACB and also ∠AOB = 2∠ADB.

5. Therefore, 2∠ACB = 2∠ADB.

6. Cancelling 2 from both sides gives ∠ACB = ∠ADB.

Hence proved: Angles in the same segment of a circle are equal.

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6. Hot Questions

1. Two chords AB and CD of a circle intersect at point E. Prove that AE × EB = CE × ED.

2. A tangent at point P of a circle touches it at P. A chord AB is drawn through P. Prove that ∠APB is equal to the angle in the alternate segment.

3. From an external point T, two tangents TA and TB are drawn to a circle with center O. Prove that TA = TB.

4. Prove that the angle between two tangents drawn from an external point is supplementary to the angle subtended by the line joining the points of contact at the center.

5. If two chords of a circle are equal, prove that the arcs subtended by them are also equal.

6. A quadrilateral is drawn inside a circle such that one of its sides is a diameter. Prove that the angle opposite to this side is a right angle.

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7. Reflective Questions and Concept Checks

- **Why does the perpendicular from the center bisect the chord?**
- **How do major and minor arcs differ in angle and length?**
- **Explain why the tangent cannot intersect the circle in two points.**
- **Prove any two radii of a circle are equal using triangle properties.**

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8. Common Mistakes & How to Avoid

Mistake	Explanation & Correction
Confusing chord with diameter	Diameter always passes through centre and is longest chord.
Assuming tangent touches circle at multiple points	Tangent touches circle at one unique point only.
Using angle at circumference as angle at centre	Centre angle is always twice the angle at the circumference.
Mixing segment with sector	Segment area excludes triangle area formed by radii; sector includes it.

9. Real-Life Applications and Fun Facts

- **Wheel design: radius defines wheel size; sector concept underlies odometer readings.**
- **Circular architecture: arches use chord and segment calculations.**
- **Ripples in water generate concentric circles.**

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10. Summary and Formulae Table

Concept	Formula / Key Fact
Circumference	$2\pi r$
Area of circle	πr^2
Arc length	$\frac{\theta}{360^\circ} \times 2\pi r$
Sector area	$\frac{\theta}{360^\circ} \times \pi r^2$
Segment area	Sector area - $\frac{1}{2}r^2 \sin\theta$
Perpendicular bisector	Bisects chord and is drawn from centre
Tangent property	Tangent $\perp$ radius at point of contact
Angle relation	Centre angle = 2 × angle at circumference

11. Study Planner and Tips

- **Day 1-2: Definitions, terms, and core theorems with proofs.**
- **Day 3: Construction methods and hands-on practice.**
- **Day 4: Important problems and solved examples.**

- Day 5: Advanced problem-solving on arcs, sectors, and tangents.
- Day 6: Review common mistakes, reflection questions, and proofs.
- Day 7: Mock test and recap with summary tables.
