CHAPTER 1: NUMBER SYSTEMS

YOUR ULTIMATE GUIDE TO MASTERING THE FOUNDATIONS OF MATHEMATICS

1. Introduction

The Number System is a fundamental concept in mathematics. It organizes numbers in a set form that helps us understand their properties and relationships. Numbers are everywhere—from counting objects to solving complex problems in science and engineering. This chapter introduces different types of numbers, their properties, and important operations on them.

2. Classification of Numbers

Numbers are classified into various types based on their properties and representations:

2.1 Natural Numbers (ℕ)

- Natural numbers are positive counting numbers starting from 1.
- Set notation: ℕ = {1, 2, 3, 4, 5, ...}

- **Properties: No zero, no negative numbers, closed under addition and multiplication.**

Example:

Operation	Result
2 + 3	5
2 × 3	6

- **Key fact: There is no largest natural number; they are infinite.**

2.2 Whole Numbers (𝕎)

- **Whole numbers include all natural numbers plus zero.**
- **Set notation: 𝕎 = {0, 1, 2, 3, 4, 5, ...}**
- **Properties: Zero included, closed under addition and multiplication.**
- **Example:**

  $4 + 0 = 4, \quad 5 \times 0 = 0$

2.3 Integers (ℤ)

- **Integers include positive and negative whole numbers and zero.**
- **Set notation: ℤ = {..., -3, -2, -1, 0, 1, 2, 3, ...}**

- **Properties: Closed under addition, subtraction, multiplication. Division is not always closed.**
- **Examples:**
  -3 + 5 = 2, -4 × 3 = -12, 7 - 10 = -3
- **Important: Division of integers can result in fractions (not integers).**

2.4 Rational Numbers (ℚ)

A rational number is any number that can be expressed as a fraction p/q, where p and q are integers and q ≠ 0.

- **Rational numbers include integers, fractions, finite decimals, and repeating decimals.**
- **Set notation: ℚ = {p/q : p, q ∈ ℤ, q ≠ 0}**
- **Examples:**
  3/4, -7/2, 0.75 (= 3/4), 0.666... (= 2/3)
- **Properties: Closed under addition, subtraction, multiplication, and division (except division by zero).**
- **Practice problem: Express 0.125 as a rational number.**
  Solution: 0.125 = 125/1000 = 1/8

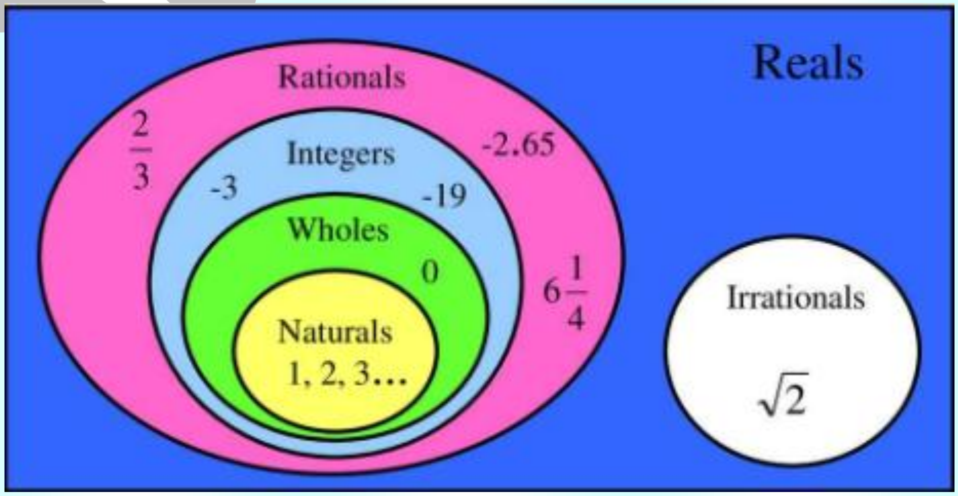
2.5 Irrational Numbers

Irrational numbers cannot be expressed as a fraction p/q. Their decimal expansions never terminate or repeat.

- Examples: $\sqrt{2} \approx 1.414213...$, $\pi = 3.1415926535...$, $e = 2.71828...$
- Key facts:
  - Between any two rational numbers, there are irrational numbers.
  - Irrational + rational = irrational.
  - Irrational × rational (non-zero) = irrational in most cases.
- **Example:**
  $1 + \sqrt{2}$ (irrational), $\sqrt{2} \times 3$ (irrational)

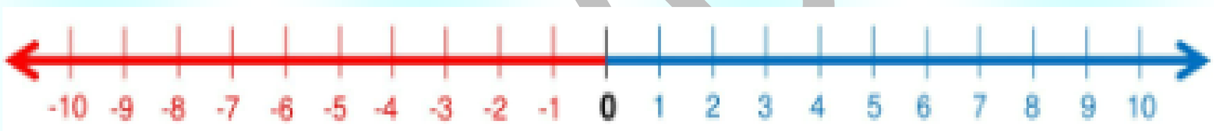
2.6 Real Numbers (ℝ)

- Real numbers include all rational and irrational numbers.
- They represent every point on the continuous number line.
- Symbolically: ℝ = ℚ ∪ irrational numbers.



3. Number Line: Visualizing Number Systems

- The number line is a straight horizontal line representing all real numbers.
- Zero is at the center; positive numbers lie right, negatives lie left.
- Points on the line correspond to real numbers uniquely.
- Rational numbers are dense on the number line; between any two, infinite rationals exist.
- Irrationals fill in gaps between rationals to make the line continuous.



4. Decimal Representation of Rational and Irrational Numbers

4.1 Terminating Decimals

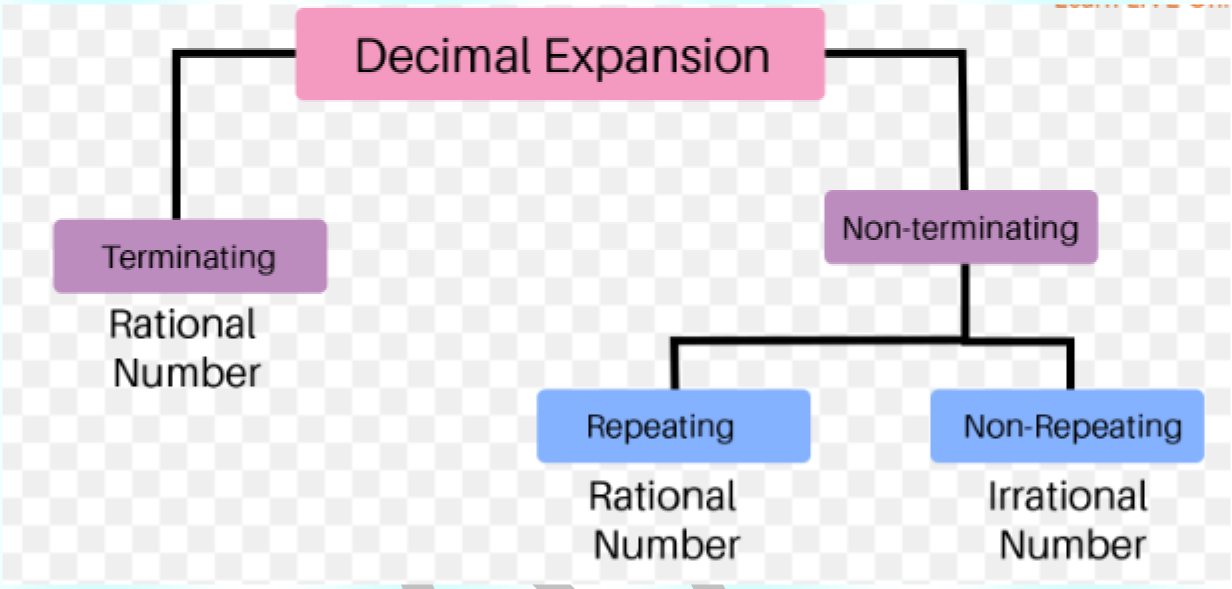
- Finite decimal expansions.
- Example: 1/4 = 0.25,   3/2 = 1.5

4.2 Non-Terminating Repeating Decimals

- Infinite decimals with repeating patterns.
- Example: 1/3 = 0.333...,   7/11 = 0.636363...

4.3 Non-Terminating Non-Repeating Decimals (Irrational)

- **Infinite decimals with no pattern.**
- **Examples: $\pi = 3.1415926535...$, $\sqrt{2} = 1.41421356...$**



5. Representation of Irrational Numbers on the Number Line

Since irrational numbers cannot be expressed exactly as fractions, we approximate their positions on the number line using geometric constructions.

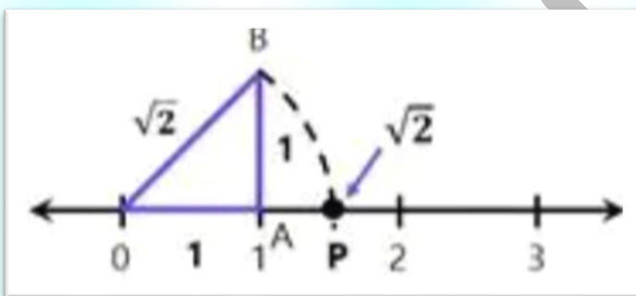
Example: Constructing $\sqrt{2}$ on a Number Line

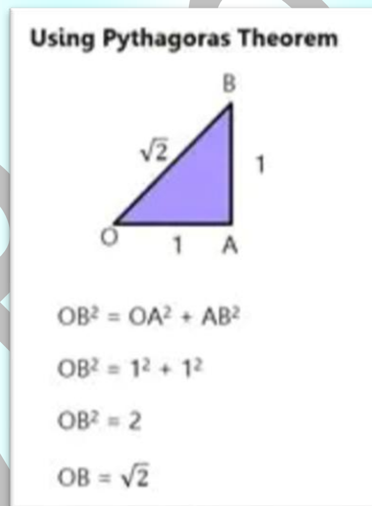
- **Draw a line segment OA = 1 unit.**
- **At point A, draw perpendicular segment AB = 1 unit.**

- Connect OB. By Pythagoras theorem,

  $OB = \sqrt{(OA^2 + AB^2)} = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

- Use a compass to transfer length OB from origin O on the number line. This point represents $\sqrt{2}$.





6. Laws of Exponents (Integral Powers and More)

For any real $a \neq 0$, and integers m, n:

- $a^m \times a^n = a^{m+n}$ (Product Rule)
- $a^m / a^n = a^{m-n}$ (Quotient Rule)
- $(a^m)^n = a^{m*n}$ (Power of Power)
- $a^0 = 1$ (Zero Power)

- $a^{-m} = 1 / a^m$ (Negative Exponent)

Example Calculations:

- Simplify: $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$
- Simplify: $5^6 / 5^2 = 5^{6-2} = 5^4 = 625$
- Evaluate: $(3^2)^3 = 3^{2\times3} = 3^6 = 729$

7. Operations on Real Numbers

- Addition, subtraction, multiplication, and division (except by zero) are closed operations within real numbers.
- Combining rational and irrational numbers must be handled carefully; the sum or product is often irrational.

8. Advanced Topics and Applications

8.1 Rationalization of Denominators

To simplify expressions involving square roots in denominators:

$3 / \sqrt{2} = 3/\sqrt{2} \times \sqrt{2}/\sqrt{2} = 3\sqrt{2} / 2$

8.2 Finding Rational Numbers Between Two Numbers

Between any two rational numbers a and b, infinite rational numbers exist. To find some:

Number between a and b = (a + b) / 2

Example: Find two rational numbers between 4 and 5:

(4 + 5)/2 = 4.5
(4 + 4.5)/2 = 4.25

9. Important Theorems

Fundamental Theorem of Arithmetic

Every composite number can be expressed uniquely as a product of primes, except for order.

Example:
$60 = 2^2 \times 3 \times 5$

10.Practice Problems and Solutions

Q1: Express 0.375 as a fraction.

Solution: 0.375 = 375/1000 = 3/8

Q2: Determine whether √5 is rational or irrational.

Solution: Irrational, as the decimal expansion is non-terminating and non-repeating.

Q3: Find a rational number between 1/3 and 1/2.

Solution:

1/3 = 0.333..., 1/2 = 0.5 → Rational between = 0.4 = 2/5

Q4: Simplify $(2^3)^4$.

Solution:

$(2^3)^4 = 2^{3 \times 4} = 2^{12} = 4096$

11.Summary and Key Points

- **Numbers are organized into different types: Natural, Whole, Integers, Rational, Irrational, and Real.**
- **Rational numbers are fractions or decimals that terminate or repeat, irrationals are decimals that never repeat or terminate.**
- **All numbers combined form the real number system, represented on the number line.**
- **Laws of exponents simplify computations.**
- **Rationalization and prime factorization are key tools for simplification and analysis.**